# Derivation of Jaynes' Sum Rule Functions
# and the
# Complex-ities of "Not"

**by**

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## Abstract

This paper provides an alternate derivation of the solution functions in Jaynes' book "Probability Theory, the Logic of Science" [1] under section 2.2, "The sum rule" [1a]. The use of differential equations is avoided entirely, without loss of generality in the end result, and the solution function for the "Not – Not(x)" problem is extended to logics of modality other than two.

The extension to modal systems is quite illuminating in that it forces a "shift of paradigm" on the view of logical negation, and illustrates a basic ambiguity in the conventional view.

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## 1) Introduction

Motivated by my own dissatisfaction in regard to transparency with the differential equation solution offered in Jaynes [1a], I attempted to derive a more intuitive solution along the same lines. An initial success in regard to discovering a symmetry requirement, and some subsequent discussion, was documented in the "concept" papers [2] and [3]. But, because of the exploratory perspective in these papers, the results probably should be considered preliminary. They are, apart from the symmetry observation, just as obscure as the original in any case.

Fortunately, an entirely different approach has proven successful. The idea is to attempt to impose general solutions that follow the form of a known solution, and find out how the constraints imposed in Jaynes restrain these general functions. The resulting, secondary, constraints then may be applied quite intuitively, at least to a scope sufficient to duplicate the solutions given in Jaynes.

In the course of the work it became apparent that the general idea behind the "Not – Not(x) = x" requirement can be generalized to modal systems, but this requires the conventional view of negation as some kind of "opposite" to be discarded in favour of a "cyclic replication" or "looped recursion" perspective. Since this result fell out of the basic development, and bears on the "sum rule" problem, I included it here.

## 2) The Basic "Not – Not(x) = x" Function Derivation

The idea of logical negation is normally taken as intuitively obvious. However, when it comes to "quantifying" the idea we would like to derive appropriate function(s) on more rigorous grounds.

One basic criterion which the bipolar (see section 11 re terminology) negation function must satisfy is the condition

$$N(N(x)) = x \qquad (2.1)$$

or, in English, "Not – Not(x) is x".

To find a suitable negation function let us assume that the basic functional form for N(x) is linear, that is

$$N(x) = ax + b \qquad (2.2)$$

Using this in (2.1) gives

$$a^2 x + ab + b = x \qquad (2.3)$$

Grouping on the basis of powers of x yields

$$x(a^2 - 1) + b(a + 1) = 0 \qquad (2.4)$$

It is tempting to clear the common factor $(a+1)$, but this would be valid only if the term is not zero. It soon will be become apparent that the term is, in fact, critical, and must be retained.

This equation must hold regardless of the value of x. Thus each of the terms in the equation must be zero independently, *i.e.*

$$a^2 - 1 = 0 \qquad (2.5)$$

and

$$b(a+1) = 0 \qquad (2.6)$$

Setting $a + 1 = 0$ satisfies both (2.5) and (2.6), and would make (2.2) become

$$N(x) = b - x \qquad (2.7)$$

This is the function commonly used for N(x), to which we will return in a moment. However, there is another solution, *viz*. setting a = 1 and b = 0. This would make (2.2) become

$$N(x) = x \qquad (2.8)$$

In English, "Not(x) is x".

At first glance this seems quite bizarre, but, in fact, it is quite meaningful. The condition we imposed, (2.1), states not only that a double negation reproduces x, but that a double “assertion” reproduces x! In philosophical terms we have the Law of Identity, “A is A”, dropping out of our imposed condition! More on this later, but it should be clear even here that we have to reconsider the basic meaning of “Not(x)”, at least from the perspective of a functional condition requirement.

Returning to solution (2.7), we need another constraint to determine the constant “b” in the general solution. That condition may be supplied by Jaynes’ equation <2.45>, repeated here (using our symbolism) for convenience.

$$x_1 N\left[\frac{N(x_2)}{x_1}\right] = x_2 N\left[\frac{N(x_1)}{x_2}\right] \qquad (2.9)$$

The task is to insert (2.7) into this to see if we can determine “b”, which will take a few steps.
I will work with only one side of (2.9), as the two sides differ by only a swap of indices. Beginning with the argument of the outer N

$$\frac{N(x_2)}{x_1} = \frac{b - x_2}{x_1} \qquad (2.10)$$

Completing the expression

$$x_1 N\left[\frac{b - x_2}{x_1}\right] = x_1\left[b - \frac{b - x_2}{x_1}\right] \qquad (2.11)$$

$$= bx_1 - b + x_2 \qquad (2.12)$$

The complete equation (adding the omitted side) thus becomes

$$bx_1 - b + x_2 = bx_2 - b + x_1 \qquad (2.13)$$

Collecting terms, and canceling the –b’s produces

$$b(x_1 - x_2) = x_1 - x_2 \qquad (2.14)$$

This is satisfied iff b = 1. That is, the r.h.s. of equation (2.7) becomes the fully defined linear function

$$N(x) = 1 - x \qquad (2.15)$$

This, of course, is the function generally accepted as the “negation” of x. However, we must still develop the other solutions shown in Jaynes <2.58> (with $m \neq 1$) to show the equivalence of the current approach. The next three sections deal with the development of these other solutions.

## 3) Generalization Based on the Linear Form

The presumption on which this section is based is that forms which reduce to a known solution are themselves solutions. For example, the well known trigonometric identity

$$\sin^2 u = 1 - \cos^2 u \qquad (3.1)$$

has the form of (2.15). If we identify "N" with the sine term, "x" with the cosine term, and solve for "N" in terms of "x" we get

$$N(x) = \sin^2 \cos^{-1} \sqrt{x} \qquad (3.2)$$

and this satisfies both of our known conditions (2.1) and (2.9). It is, in fact, the same solution as (2.15), although this is not conspicuous.

Let us use the same principle, but introduce arbitrary functions into (2.15) and then determine constraints on these functions that are equivalent to our known conditions.

$$f(N) = 1 - g(x) \qquad (3.3)$$

Assuming f(N) can be inverted, that is, if $f(N) = c$ then we can solve for N to get

$$N = f^{-1}(c) \qquad (3.4)$$

equation (3.3) can be rearranged to give

$$N = f^{-1}(1 - g(x)) \qquad (3.5)$$

Applying our first constraint, (2.1), then gives

$$N(N(x)) = f^{-1}[1 - g\{f^{-1}(1 - g(x)\}] \qquad (3.6)$$

If f(x) and g(x) are the same function (3.6) can be reduced, as follows

$$N(N(x)) = g^{-1}[1 - (1 - g(x))] \qquad (3.7)$$

$$= x \qquad (3.8)$$

and our first condition is satisfied. (I suspect the equivalence of f(c) and g(c) is the symmetry constraint illustrated in [2], but I haven't confirmed it.)

Equation (3.3) for further work has become

$$g(N) = 1 - g(x) \qquad (3.9)$$

or, getting N out explicitly,

$$N(x) = g^{-1}(1 - g(x)) \qquad (3.10)$$

At this point there is no reason to assume $g(x)$ or it's inverse is distributive over addition, so this is as far as we can reduce the r.h.s.

We proceed by using this result in our second constraint, (2.9). Again, I will work with one side only, exploiting the obvious symmetry to simplify the development. We have

$$x_1 N\left[\frac{N(x_2)}{x_1}\right] = x_1 N\left[\frac{g^{-1}(1-g(x_2))}{x_1}\right] \qquad (3.11)$$

$$= x_1 g^{-1}\left(1 - g\left[\frac{g^{-1}(1-g(x_2))}{x_1}\right]\right) \qquad (3.12)$$

To reduce this further we need to require some form of distributive law. There are at least three possibilities.

## 4) Geometrically Distributive Solution

Ordering the three distributive forms arbitrarily, the first is

$$g\left(\frac{a}{b}\right) = \frac{g(a)}{g(b)} \qquad (4.1)$$

Applying this to the g with the argument in square brackets in (3.12) yields

$$x_1 N\left[\frac{N(x_2)}{x_1}\right] = x_1 g^{-1}\left(1 - \frac{1-g(x_2)}{g(x_1)}\right) \qquad (4.2)$$

$$= x_1 g^{-1}\left(\frac{g(x_1) - 1 + g(x_2)}{g(x_1)}\right) \qquad (4.3)$$

Adding the r.h.s. of the complete equation and taking $x_1$ and $x_2$ to the opposite sides

$$\frac{g^{-1}\left(\frac{g(x_1)-1+g(x_2)}{g(x_1)}\right)}{x_2} = \frac{g^{-1}\left(\frac{g(x_2)-1+g(x_1)}{g(x_2)}\right)}{x_1} \qquad (4.4)$$

Taking g of both sides, using (4.1) again, then taking the denominators to the opposite sides leaves

$$g(x_1) - 1 + g(x_2) = g(x_2) - 1 + g(x_1) \qquad (4.5)$$

The left and right sides of our starting equation (2.9) have become identical.

Trying a few obvious candidates for g(c) to find one that satisfies (4.1) we soon locate the exponential form

$$g(c) = c^m \qquad (4.6)$$

which, when entered in (3.10) gives us Jaynes' solutions <2.58>, repeated here for convenience.

$$N(x) = (1 - x^m)^{\frac{1}{m}} \quad (4.7)$$

Equation (4.6) is, of course, a guess, but it is a simpler guess than that used as the final step in getting <2.58> from <2.57> in Jaynes.

## 5) Arithmetically Distributive Solution

A second possible form of distributive law is

$$g^{-1}(a+b) = g^{-1}(a) + g^{-1}(b) \quad (5.1)$$

Applying this to (3.12) gives

$$x_1 N\left[\frac{N(x_2)}{x_1}\right] = x_1\left(g^{-1}(1) - \frac{g^{-1}(1) - x_2}{x_1}\right) \quad (5.2)$$

$$= x_1 g^{-1}(1) - g^{-1}(1) + x_2 \quad (5.3)$$

Adding the right hand side of the original equation and collecting terms gives

$$g^{-1}(1)x_1 - g^{-1}(1) + x_2 = g^{-1}(1)x_2 - g^{-1}(1) + x_1 \quad (5.4)$$

$$g^{-1}(1)(x_1 - x_2) = x_1 - x_2 \quad (5.5)$$

Since $x_1 \neq x_2$ in general this implies

$$g^{-1}(1) = 1 \quad (5.6)$$

Taking $g$ of both sides gives

$$1 = g(1) \quad (5.7)$$

Using this in (3.9) we get

$$g(N) = g(1) - g(x) \quad (5.8)$$

and, applying $g^{-1}$ to both sides, then using the (second) distributive law (5.1)

$$N(x) = 1 - x \quad (5.9)$$

which is our original, linear, solution, (2.15). It is interesting that this common form of distributive law doesn't lead to any other possible solutions.

## 6) Another Geometrically Distributive Solution

A third possible form of distributive law is

$$g\left(\frac{a}{b}\right) = g(a)g(\frac{1}{b}) \qquad (6.1)$$

(This is the same as the first distributive law iff $g\left(\frac{1}{b}\right) = \frac{1}{g(b)}$ )

Applying this to (3.12) we get

$$x_1 N\left[\frac{N(x_2)}{x_1}\right] = x_1 g^{-1}\left[1 - g\left(\frac{g^{-1}(1-g(x_2))}{x_1}\right)\right] (6.2)$$

$$= x_1 g^{-1}\left[1 - [1 - g(x_2)]g\left(\frac{1}{x_1}\right)\right] (6.3)$$

$$= x_1 g^{-1}\left[1 - g\left(\frac{1}{x_1}\right) + g\left(\frac{x_2}{x_1}\right)\right] (6.4)$$

Adding the r.h.s. of the complete equation and taking $x_1$ and $x_2$ to the opposite sides

$$\frac{g^{-1}\left[1 - g\left(\frac{1}{x_1}\right) + g\left(\frac{x_2}{x_1}\right)\right]}{x_2} = \frac{g^{-1}\left[1 - g\left(\frac{1}{x_2}\right) + g\left(\frac{x_1}{x_2}\right)\right]}{x_1} \qquad (6.5)$$

Take g of both sides and use the (third) distributive law to get

$$\left[1 - g\left(\frac{1}{x_1}\right) + g\left(\frac{x_2}{x_1}\right)\right] g\left(\frac{1}{x_2}\right) = \left[1 - g\left(\frac{1}{x_2}\right) + g\left(\frac{x_1}{x_2}\right)\right] g\left(\frac{1}{x_1}\right) \qquad (6.6)$$

Using the (third) distributive law twice again and canceling the middle terms produced the final result

$$g\left(\frac{1}{x_2}\right) + g\left(\frac{1}{x_1}\right) = g\left(\frac{1}{x_1}\right) + g\left(\frac{1}{x_2}\right) \qquad (6.7)$$

which is an identity. That is, all functions that obey the third distributive law are solutions to both our basic conditions. The identity operator, N(x) = x, is a simple example. The solution using the first distributive law, (4.6), is also a solution here.

At this point we have arrived at a position equivalent to that in Jaynes. We have the basic solution, (2.15), and the power solution, (4.7) (by two different developments). I will leave the reader to decide which approach is more transparent.

## 7) Extension of the Cyclic Replication Criterion to Modal Systems

So far we have been dealing with a bipolar system, in which a double application of the negation function reproduces the original argument. It is quite instructive to extend this to systems of other polarities.

In a unipolar system there is only one logical value, Existence (note the capital E), so the only function possible is N(x) = x. This is the Law of Identity, “A is A”.

The same result appeared unexpectedly in the bipolar analysis (area around equation (2.8)), but it is not an accident from a conceptual standpoint. In essence the “looped recursion” equation expresses a repeated “assertion” condition as well as repeated negation. Furthermore, the assertive meaning is the “base” solution of the equation, in all polarities; even in the unipolar case where the negation meaning cannot apply.

To take any value something must exist that is capable of taking values, and the equation forces this upon us as the solution common to the equations for all polarities.

To illustrate this further, consider a tripolar system, in which we generalize the form to three applications

$$N(N(N(x))) = x \qquad (7.1)$$

Assuming a linear form for N(x) again, (7.1) becomes

$$a[a(ax+b)+b]+b = x \qquad (7.2)$$

This reduces to

$$x(a^3-1)+b(a^2+a+1)=0 \qquad (7.3)$$

Since x is variable and (7.3) must apply under all conditions each term must be zero independently. That is

$$a^3 = 1 \qquad (7.4)$$

and

$$b(a^2+a+1)=0 \qquad (7.5)$$

Considering (7.4), the solutions are

$$a = 1, e^{j2\pi/3}, e^{-j2\pi/3} \qquad (7.6)(7.7)(7.8)$$

where j is the root of –1 and e is the base of natural logs.

The first solution, (7.6), is the Law of Identity again. Interestingly, this solution for “a” also forces “b” to 0 to satisfy (7.5), thus preventing the “dilution” of the Law of Identity.

Solutions (7.7) and (7.8) are +120 and –120 degree rotations respectively of the unit “identity” vector in the complex plane. Clearly, three successive applications (the system is tripolar) of either will return the starting argument value. Each of these solutions for “a” will force the coefficient of “b” in (7.5) to zero, and thus will permit “b” to be chosen by other criteria.

This tripolar result clarifies the overall situation. That is, the conventional "negation" should not be considered some kind of "opposite", but one state in a sequence of "cyclic replication" or "looped recursion". The number of states is expressed as the modality of the system in question. With this meaning we get the Law of Identity in every modal system by the above definition process, and have a completely general understanding of the meaning of "negation" across systems of all modalities.

This perspective also shows that a bipolar system confuses of two kinds of "opposites" as one "negation", *viz*. the complement (a union of all other values, or absence of a given value) and the reversed vector. There is no reversed vector in the tripolar system, or in any system with odd modality. In bipolar both meanings yield the same result, raising inferential hazards in regard to equivocation and ambiguity wherever the distinction is important.

The solution for a general system of polarity "n" is

$$x(a^n - 1) + b\sum_{i=1}^{n-1} a^i = 0 \qquad (7.9)$$

The "Law of Identity" solution (a = 1, b = 0, N(x) = x) is a solution for all "n". For all other solutions the summation is a vector sum whose resultant is zero, leaving "b" free to be set by another constraint. The solution is completely general.

The question of interpretation of the vector model deserves some attention. In reference [5] I used a model of logical and/or physical systems based on matching the total number of states in the real world system with a combination of degrees of freedom and number of states for each in a logical model. For example, a time would be represented by hours, minutes and seconds; for three degrees of freedom. The "hours" variable has a range of values 0 through 23, for a modality of 24, and so on. The total number of states would be 24*60*60 = 86,400.

In the current vector model each degree of freedom would be a vector. The magnitude could encode the probability of the existence, or perhaps the certainty that the degree of freedom is correctly identified. The direction would indicate the current modal state. The two ideas are conceptually distinct, and should really be represented in the mathematics in a way that is meaningful across all modalities. The results in this paper indicate how that might be done.

A search of the internet came up with only one paper that applies complex numbers to logical problems [4], but the application there was to model (hidden) ambiguous behaviour in logic gates, and to apply this to model quantum mechanical phenomena.

## 8) Summary and Conclusions

When I began this intellectual journey I had no idea that the use of recursive functions would lead where it has. There were many surprises. To wit:

The natural appearance of complex plane theory was totally unexpected. I have long thought that there was some potential for complex numbers in logic, but the natural appearance here was not anticipated.

The Law of Identity issue in particular was a real shock. To find the mathematics indicating that assertion and negation were both solutions of the same requirement seemed totally bizarre when it arose, but the "paradigm shift" in thinking that eventually fell out was a true expansion of insight, at least for me. Negation will never be the same!

It was a comparable surprise to find the complex plane model mapping onto the ideas expressed in [5]. The idea of having, and needing, separate probabilities for existence and modality is certainly new to me. Yet, given the general modeling structure, it now seems obvious. They are, in reality, separate questions and mathematical descriptions really should encode them separately.

The complex plane model made the conceptual split of existential and modal probabilities necessary, and seems to map the split very well. I will be interested to see what develops.

A natural extension of the vector idea would be to permit vector addition, defining resultant and/or reversed vectors that do not lie along the modal directions. The algorithm(s) defining the rules of addition would have to yield magnitudes restricted to one or less, which suggests the forms of entropy calculations as candidates.

Simplification of Jaynes' "Sum Rule" development has to be progress. I sincerely regret having to take a position critical of the original on the point, but I trust my comments will be taken as they are intended, with great professional respect for the achievements the original represents in context.

There is some concern over using continuous mathematics in combination with discrete modal ideas. Intuitively, I suspect techniques such as the Z transform of digital theory, or difference equations might prove relevant, but the question ultimately deserves a rigorous examination, and clarification of the precise relationship.

## 9) Acknowledgments

The ideas presented above were very much a journey of discovery for the author, but the interest would not have developed without the efforts of E. T. Jaynes and the people who are now engaged in perpetuating and extending his contributions. In particular I would like to express my appreciation to the editor of [1], Larry Bretthorst, for bringing the book to publication, and to Dr. Kevin Van Horn, whose web site led me to Dr. Jaynes' work, and whose patience and assistance in my struggles with technical problems were invaluable. I can only hope the above ideas will provide some reward for them all for their efforts, especially the efforts on my behalf.

## 10) Bibliography

[1] Book: "Probability Theory: The Logic of Science", E. T. Jaynes, Edited by G. Larry Bretthorst. Cambridge University Press, 2003

[1a] Section 2.2 "The Sum Rule", pp30 *et seq.*

[2] Paper: JaynesCritiqueA.pdf, Alan G. Herron. Available at www.greatblue.ca (same site as this paper)

[3] Paper: JaynesCritiqueB.pdf, Alan G. Herron. Available at www.greatblue.ca

[4] Paper: A Complex Logic for Computation with Simple Interpretations for Physics. Richard G. Shoup. Interval Research, Palo Alto, CA 94304. Contact (as quoted in paper): shoup@interval.com

[5] Paper: SharpeningOccamsRazor.pdf, Alan G. Herron. Also available at www.greatblue.ca

## 12) Addendum: A Few Words on Terminology

There does not appear to be a uniform vocabulary for the various similar ideas in logics, which can lead to equivocation, confusion and communication difficulties. Consider, for example, these ideas:

"valence" the number of arguments an operator takes. Because this is often two the operators are often referred to as binary operators, even if the arguments are modal. The operator "output" may take on more than two values in a modal system even though the operator only takes two arguments. The term "valence" does not appear in the literature to my knowledge, but the idea is different from the number of values an argument or function may take, and should be distinguished.

| | |
|---|---|
| “binary” | a system with exactly two discrete values. If the system is extended to the continuous domain these become the limits or extremes of the range, and are still referred to, sometimes, as binary even though the idea is then “bipolar”. |
| “bipolar” | any system with exactly two extreme values, including binary and a continuous range with exactly two extremes. Clearly the extension to continuous variable values makes this different from “binary”, which, strictly speaking, has only two discrete truth-values. |
| “modal” | a system with a number of discrete values other than two, usually more. Again one can extend this to a statistical view wherein the modal values are the “poles” of continuous ranges. The idea of an operator that takes more than two arguments (i.e. modal-valent) is another possible meaning. |
| “modal-polar” | the analog of bipolar for modal systems. |
| “Boolean” | the system defined by George Boole, including his binary values and choice of operators. I understand the “or” in Boole’s original work is exclusive, and the system thus differs from common binary logic wherein the “or” is inclusive. Both are often called Boolean logics, presumably because they both deal with binary variables. |

etc.

I hope the meaning of the terms I use is clear in context. Without standardized vocabulary it is difficult to express the ideas with more precision.

AGH