Domain Limits and Functional Forms: Clarification and Extension of E. T. Jaynes' Results

by

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2004 April 17

Abstract

This monograph gives alternate, but parallel, derivations to Section 2.2 "The Sum Rule" in E. T. Jaynes' book "Probability Theory: The Logic of Science". Domain limits are reworked more carefully for the derivations on pages 30 and 31, and relevant supplementary material is provided to make the argument more transparent. In the process the development for the form of S(U) on pages 32 through 33 is made essentially redundant.

Part 1: Re-Derivation of the Development Up to Equation 2.46 Inclusive

Equation 2.36 is given as

$$V = S(U) \quad (1a)$$

$$0 <= U <= 1 \quad (1b)$$

Equation 2.45 is given as

$$X \; S[S(Y) / X] = Y \; S[S(X) / Y] \quad (2a)$$

with domain limits

$$0 <= S(Y) <= X \quad (2b)$$

$$0 <= X < = 1 \quad (2c)$$

The argument of S(U) has a range limit given by (1b). Looking at the outer of the nested functions S[…] in 2a thus gives us

$$0 <= S(Y) / X <= 1 \quad (3a)$$

or, multiplying by X

$$0 <= S(Y) <= X \quad (3b)$$

which is Jaynes' limit (2b), and

$$0 <= S(X) / Y <= 1 \quad (4a)$$

or, multiplying by Y

$$0 <= S(X) <= Y \quad (4b)$$

which is not included explicitly in the text.

By the same argument on the inner of the nested S functions

$$0 <= Y < = 1 \quad (5)$$

and

$$0 <= X < = 1 \quad (6)$$

Of (5) and (6) only (6) is included in the text as (2c). We do not wish to limit Y any more than X, and we wish to preserve symmetry, so, intuitively, it would seem the auxiliary domain constraints (4b) and (5) should accompany equation (2a)

It thus appears that Jaynes' domain limits are incomplete. Furthermore, it is not obvious that they are totally compatible, given the simultaneous necessity of functional relations (3b) and (4b).

The problem here is similar to dealing with boundary value problems in differential calculus, often approached using Green's functions. We want to restrict the function S(U) with known constraints without totally specifying it. In other words, the freedom in the connection of X and Y in (3b) and (4b) is in the specific functional form of S(U). Both X and Y are (at least in our desires) still independent, free variables.

Moving on to the derivation of 2.46 from 2.45…

Jaynes makes the bridge by considering the special case Y = 1. This is open to some objection in that it restricts Y from being a free variable as an essential step of the derivation. Is it possible to get to 2.46 another way?

Consider (2a). We want both X and Y to remain free and independent. To satisfy (2a), therefore, along with the requirement that 0 <= S(U) <= 1, the only way (2a) can be satisfied is for

$$S[\ S(Y) / X\ ] = 0 \quad (7)$$

and

$$S[\ S(X) / Y\ ] = 0 \quad (8)$$

With these conditions equation (2a) becomes X * 0 = Y * 0, which leaves both X and Y as free variables, with completely symmetrical constraints.

We know from our general constraints that

$$S[1] = 0, \quad (9)$$

Jaynes requires S(U) to be a continuously decreasing, monotonic function. Thus U = 1 is the only argument which makes S(U) = 0.

Comparing this with (7) gives

$$S(Y) / X = 1 \quad (10a)$$

or

$$S(Y) = X \quad (10b)$$

Similarly, from (8)

$$S(X) / Y = 1 \tag{11a}$$

or

$$S(X) = Y \tag{11b}$$

Comparing (10b) with (3b) shows the latter is too broad. The constraint (11b) is missing entirely in the text.

Both arguments in (7) and (8) potentially contain divisions by zero. Our new constraints, (10b) and (11b), block any difficulties which might arise from this hazard.

Substituting Y from (11b) in (10b) gives

$$S(S(X)) = X \tag{12}$$

without using any special case constraint.

The derivation of (2b) given from 2.46 through 2.47 *et seq* now becomes redundant as well. It remains interesting from the point of view of conceptual integration.

Part 2: Derivation of the Form of S(U)

The problem now addressed is the interpretation of (10b) and (11b), and the implications to the specific functional form of S(U). Jaynes, on pp 32 *et seq* , goes through an extensive derivation and solution of a differential equation, using binary and Taylor expansions, exponential substitutions, arguments concerning taking limits, and so on. The result he derives is actually too narrow, and, with an alternate perspective, the general solution becomes intuitively obvious.

The essential property of S(U) is self-replication. That is, whatever function S(U) is, it must satisfy equation (12) above. Let us look at the geometry of a system that satisfies this condition.

Consider Figure 1

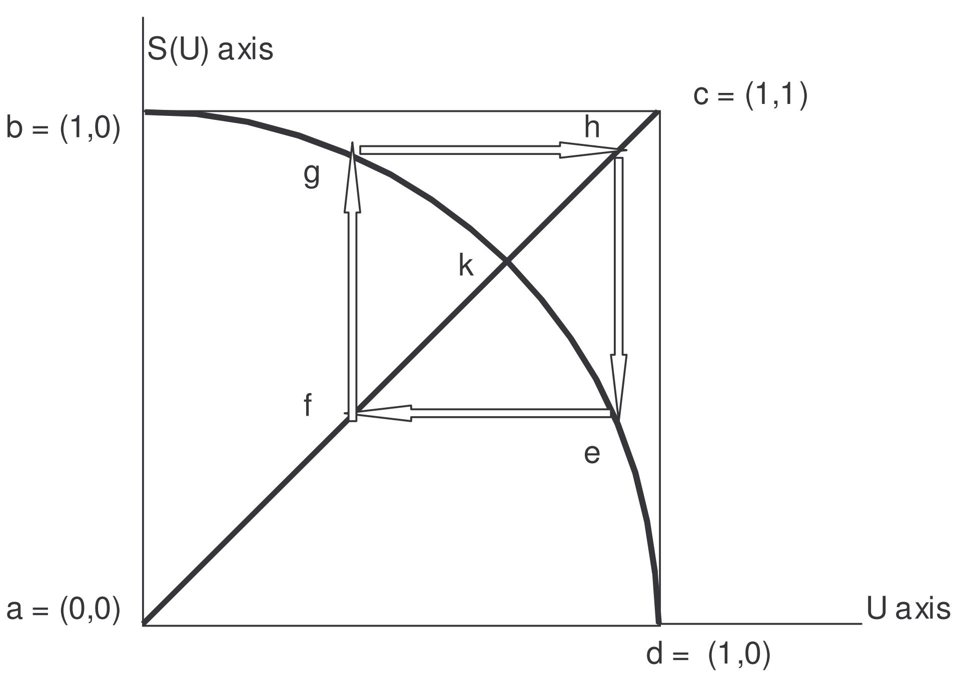

Figure 1: Self-Replicating Geometry

abcda is the unit square, which is the domain of concern. Line afkhc is a diagonal of this square. Algebraically it is denoted by the equation

$$S(U) = U \qquad (13)$$

The curve bgked is the function S(U). It is drawn here as a circular arc, but is a general curve which meets Jaynes' specifications, *viz*

$$S(0) = 1 \qquad (14a)$$

$$S(1) = 0 \qquad (14b)$$

and is decreasing continuous and monotonic. (14c)

To begin, start at point e on S(U). This has an abcissa Ue, and an ordinate S(Ue). We want to use S(Ue) as an argument, so we follow the arrow to point f, which is on the line S(U) = U, and thus gives us our new Uf. We then move vertically (i.e. Uf = Ug) to point g, where S(U) = S(Ug).This gives us the value of our desired function S(S(Ue)). To see what this is on the U axis follow the horizontal arrow to the right to point h, where the value of U is Uh.

To satisfy our self-replicating requirement Uh must be the same as Uf. That is, our traverse must be a closed square, efghe.

In order for this to be true the function S(U) must be symmetric about the diagonal. That is the only requirement on the shape necessary to guarantee that the traverse closes.

The coordinates of our starting point are

$$Pe = (Ue , S(Ue)) \qquad (15)$$

The coordinates of the symmetrical point on the curve are

$$Pg = (Ug , S(Ug)) \qquad (16)$$

To be symmetrically placed around the diagonal the U and S(U) coordinates of the points must "cross" roles, that is

$$Ue = S(Ug) \qquad (17a)$$

$$Ug = S(Ue) \qquad (17b)$$

If you now look at equations (10b) and (11b) you will see this is precisely what they say. In other words, the constraint (2a) is guaranteed to generate a self-replicating function S(U) by the constraints we have supplied.

It is easy to visualize functions which are symmetrical about the diagonal (satisfy 17a..b) and meet the other criteria (14a..c). A couple of straight lines, dkb, work. A cosine function with the arguments "rotated" 45 degrees should do it. A Fourier series of cosines should work. A rotated parabola should work. Jaynes' S(U) = 1 - U works (this is the other diagonal of the square, obviously symmetric). I used a circular arc in the drawing, which should be quantitatively valid. Boolean (Aristotelian) logic is edge pairs of the outer square itself, approached as a limit(s) of the two lines dkb, with k pushed into the corner(s) (0, 0) or (1,1).

Jaynes' general solution satisfies the constraint, but appears more restrictive than the general "even function" constraint developed here.

To see this let us rotate a conventional (x,y) coordinate frame 45 degrees clockwise into a new frame (u,v) [note these variables have no relation to the above notation]. The transformation equations are

$$u = (x - y) / \mathrm{sqrt}(2) \qquad (18a)$$

$$v = (x + y) / \mathrm{sqrt}(2) \qquad (18b)$$

Inverting gives

$$x = (u + v) / \mathrm{sqrt}(2) \qquad (19a)$$

$$y = (-u + v) / \mathrm{sqrt}(2) \qquad (19b)$$

Jaynes' general solution, 2.58, substituting X = x and S = y, and reworking slightly, is

$$x^m + y^m = 1 \qquad (20)$$

Substituting from (19a..b) to see what this looks like in the rotated frame

$$(u + v)^m + (-u + v)^m = 2^{(m / 2)} \qquad (21)$$

This is an even function. That is v(u) = v(-u), for, if we substitute u -> -u

$$(-u + v)^m + (u + v)^m = 2^{(m / 2)} \qquad (22)$$

which is the same as equation (21) with the terms interchanged.

It is obvious that there are even functions that do not satisfy Jaynes' form with a single m, so his solution appears less general than the above development implies. See the discussion for further comment.

I suggest that the above derivation is a lot simpler and more intuitive than that in the text. Perhaps it is too much to claim the development on pages 32 and 33 is now redundant, but the text is very heavy going if the above actually gets us to the same (or better) place.

Discussion

The symmetry about the diagonal constraints are fairly easy to derive formally. The process is exactly parallel to the demonstration of the general compatibility of Jaynes' function 2.58, with a slight deviation in the final stages. I would have done it here, but it gets a bit tedious and is not really essential to the general argument. The point is that the symmetry requirement can be derived rigorously with little trouble if necessary.

An interesting question arises in regard to choosing which of the plethora of functions that pass all our tests to use. It occurs to me that this choice might be related to entropy issues. I will digress for a moment here to explain.

Referring to Figure 1, a curve S(U) crosses the main diagonal at the point k where U is equal to S(U). If this is not in the centre of the square, as it is for the conventional S(U) = 1 – U, the value of U is not a half. Under equal probability conditions for U and S(U) the value of a half represents the point of maximum entropy. Yet the general point, k, might well represent a "unbiased" or "most expected" situation in a given real problem. In other words, the highest entropy situation appears to be defined or related to the indicated crossing.

In any case it would be very interesting to know what the crossing point signifies, or, more generally, the choice of function. I suspect all curves aren't born equal in some meaningful regard from a modeling viewpoint. Solid conceptual control and use of the resulting flexibility in solutions should make setup and interpretation of results in real world problems more intuitive. That has to be a desirable feature of any theory.

Jaynes' (Cox's) general forms appear to be more restrictive than necessary. It is a matter of academic interest to determine where the additional restriction was introduced in the derivation, or, if the derivation is valid, where the excess generality occurs in the foregoing development. It might have something to do with the differences in the domain determinations, or…

It might be possible to sum an infinite series of Jaynes' functions, along the lines of a Fourier or Legendre series expansion, allowing m to take on successive integer values and summing the results, to generate an arbitrary even function. In other words, Jaynes' solution may be a series solution for the general set of even functions. This particular series isn't familiar to me (does anyone recognize it?), but the form is very similar to other series solutions of differential equations. If this is the case, then Jaynes' (Cox's) solution is completely general, but in a rather obscure form.

Finally, it is interesting to examine a few special cases of Jaynes' forms, (20).

If m is negative we get into trouble at U or S(U) = 0. These solutions appear to be appropriate for the region outside the one of current interest, that is 1 < u, S(U). "Even" functions in general are not restricted to 0 <= y <= 1, so the general solution to the differential equation likely generated them. This automatic "symmetry" of the mathematics is quite amazing.

If m = 1, we get conventional continuous probability, S(U) = 1 – U.

If m = 2, S(U) becomes the arc of a circle centred on the origin, the one I used in Figure 1.

If m becomes greater than 2 the functions become the (very interesting and attractive) curves developed by Piet Hein of Sweden called super eggs. There is a traffic roundabout in Copenhagen whose plan was designed to one of these curves. Perhaps the relationship has something to do with the low statistical frequency of accidents in the facility :-)

And, if m goes to infinity we get the square abcda, which is the Boolean case discussed as the limit of the double line, dkb, solution.

Conclusion and Acknowledgments

I sincerely hope this saves someone some of the effort I had to expend in understanding what Jaynes was up to. I also hope none of the above will be taken as critical of Jaynes, or his editor G. Larry Bretthorst. I found the conceptual outline of the process in the text quite exciting and inspiring, and it would please me enormously if my efforts make the original work more open to others with an interest in the subject. I would not have come to any understanding of the subject without the original text to serve as a guide and inspiration.

I would be seriously remiss if I did not express my thanks to Kevin van Horn for his dedication, time, and patience in managing the web site and helping me personally through the development

in the text. We have a somewhat different perspective on a few technical points, which undoubtedly ran up his stress levels a bit, but he endured. Thanks Kevin. You were a big help.

For those with continued interest I would refer you to the unofficial errata and discussion web site. You will find access to Kevin's work, user forums, and related links there. The URL is

http://leuther-analytics.com/jaynes/

In reading Jaynes, without fully understanding his arguments or assumed background, it was sometimes difficult to determine precisely what was his intent, and the general drift of the argument. Kevin van Horn tried to help here, but if I've missed the point, or if Jaynes actually was making the same argument in different words he should get the credit.

Finally, the intent of this document is not to present a formal paper for the professional community, so I played it a bit "loose" in regard to background research and so on. If I have not given due credit, or replicated the works of others I apologize. The work was original in the sense I didn't copy it, but I am, perhaps, not the first to reach the views expressed herein. If so I would be happy to provide links or references as an appendix to this document on being so advised. I am a retired engineer, with a long-standing personal interest in epistemology. This is recreation for me, not professional activity.

And now, on to the adventure of the rest of the text…
